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A New Algorithm for Drawing Series-Parallel Digraphs in 3D

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Abstract

This paper proposes a new algorithm for drawing series-parallel digraphs in three dimensions. Our algorithm produces a three dimensional strictly upward Fary grid drawing with volume $O(n^3)$ for an arbitrary series-parallel digraph. We also prove that if series-parallel digraph is regular, it can be drawn with volume $O(n^2)$ and further, if a series-parallel digraph is regular and its structure tree fulfils a simple condition, the graph can be drawn inside a box having volume $O(n)$.

1 Introduction

Three dimensional drawings of graphs are needed in VLSI-design, modeling VRML worlds and in user interface design. There is also a great theoretical interest to learn properties of the three dimensional drawings of graphs.

The first theoretical results for the three dimensional graph drawing problem appeared in [4] where it was proved that a complete graph with n vertices has a three dimensional Fary grid drawing with volume $2n \times 2n \times n$. It was conjectured that there are other classes of graphs allowing a smaller volume than the general case.

The next refinement was given in [2], where it was shown that 2-, 3- and 4-colorable graphs can be drawn with volume $O(n^2)$ and that the lower bound for their drawing is $O(n^{3/2})$. In [12] it was pointed out that any C-colorable graph can be drawn with volume $O(n^2)$. Further results appeared in [7], where it was proved that 2- and 3-colorable graphs can be drawn with volume $O(n^{3/2})$ and that any C-colorable graph admits a drawing with volume $O(C^4 n^{3/2})$.

For series-parallel digraphs a three dimensional drawing algorithm appeared in [8], but no concrete results was given for the volume of the obtained drawing. An algorithm based on producing first a two-dimensional drawing and then rotating it to obtain a three dimensional drawing.

In this article we describe an algorithm that directly produces three dimensional drawings for series-parallel digraphs. Our drawing method has similarities with $\triangle$-algorithm [1, 3], which produces two-dimensional drawings for series parallel digraphs. We show that series-parallel digraphs can be drawn with volume $O(n^3)$ and that there are some classes of series-parallel digraphs that can be drawn with volume $O(n^2)$ and even with volume $O(n)$.

2 Preliminaries

For the basic graph-theoretical concepts we refer to [13] and for algorithms and their notations to [6]. Notations and definitions for the three dimensional graph drawing are mostly taken from [4, 2] and for series-parallel digraphs from [1, 3, 14].

The *Fary grid drawing* of a graph is a three dimensional drawing where vertices are placed at integer coordinates, edges are straight-lines and crossings of edges are not allowed. A drawing of an acyclic digraph is *upward*, if each edge is drawn as a curve monotonically nondecreasing in the predefined direction. It is *strictly upward*, if each edge is drawn as a curve strictly increasing in the predefined direction.

Let $\mathcal{D}$ be a three dimensional drawing. The *rectangular hull* of $\mathcal{D}$ is the smallest rectangular prism with sides parallel to coordinate axis containing the whole drawing. The *volume* of $\mathcal{D}$ is the product of the lengths of the three sides of the rectangular hull of $\mathcal{D}$. The *footprint* of a three dimensional drawing is its projection on the xy-plane.

A *source* of a digraph is a vertex without incoming edges and a *sink* is a vertex without outgoing edges. An edge (v, u) is *transitive* if there is directed path from v to u such that (v, u) doesn't belong to that path.

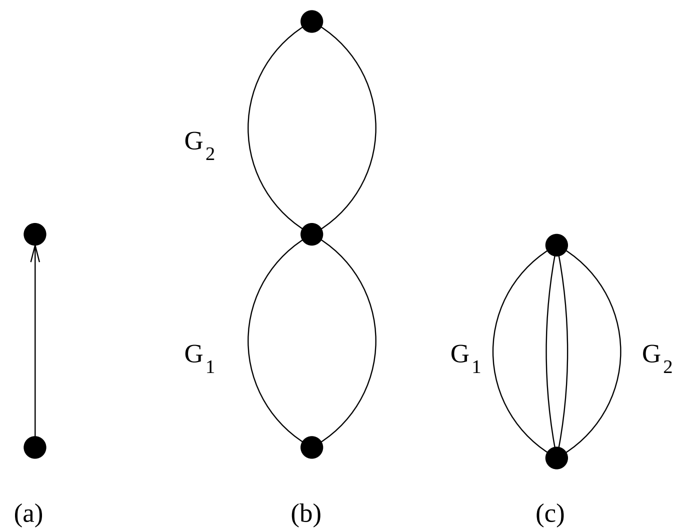

Figure 1: Recursive definition of a series-parallel digraph: (a) base case, (b) series composition, (c) parallel composition.

A *series-parallel* digraph is recursively defined as follows:

1. A digraph consisting of two vertices and an edge joining them is a series-parallel digraph.

2. If $G_1, \ldots, G_k$ are series-parallel digraphs so are the digraphs obtained by the following operations:

(a) The series composition of digraphs $G_1, \dots, G_k$ with sources $s_1, \dots, s_k$ and sinks $t_1, \dots, t_k$ is the digraph obtained by identifying the sink t_i with the source s_{i+1} where $1 \le i < k$.

(b) The parallel composition of digraphs $G_1, \dots, G_k$ with sources $s_1, \dots, s_k$ and sinks $t_1, \dots, t_k$ is the digraph obtained by identifying $s_1, \dots, s_k$ into a single vertex s and identifying $t_1, \dots, t_k$ into a single vertex t.

Throughout this paper, we assume that there is no parallel edges. A series-parallel digraph G is associated with a rooted tree T, called *SPQ-tree* or *decomposition tree* [14, 1, 3]. There are three types of nodes (S-, P- and Q-nodes) in a decomposition tree:

1. If G is single edge, then T consists of a single Q-node.
2. If G is created by the parallel composition of series-parallel digraphs $G_1, \dots, G_k$ with decompositions trees $T_1, \dots, T_k$, then the root of T is a P-node and its children are subtrees $T_1, \dots, T_k$.
3. If G is created by the series composition of series-parallel digraphs $G_1, \dots, G_k$ with decompositions trees $T_1, \dots, T_k$, then the root of T is an S-node and its children are subtrees $T_1, \dots, T_k$.

   See Figure 2 for an example of a series parallel digraph and its decomposition tree.

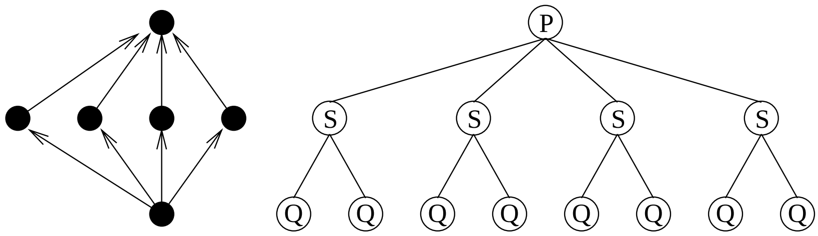

Figure 2: A Series-parallel digraph and its decomposition tree.

If G has n vertices, then T has $O(n)$ nodes. T can be constructed in $O(n)$ time using the recognition algorithm introduced in [14]. The recognition algorithm produces a binary decomposition tree, which is easy to modify [3] to a tree for which the following two invariants holds:

- If the type of a node is P, then all children are S-nodes except that there might be one Q-node. If there is a Q-node, then it is a transitive edge.
- If the type of node is S, then all children are P- or Q-nodes.

If these two conditions holds, we call the decomposition tree a *structure* tree.

The structure tree T of a series-parallel digraph G is *symmetric* if, for each node v in T, it holds that if v_i and v_j are children of v, then subtrees T_i and T_j with roots v_i and v_j coincide. If a series-parallel digraph G has a symmetric structure tree, then we say that G is *regular*.

In this article we use simultaneous Cartesian coordinate system and *spherical coordinate system*. The spherical coordinates of a point p are (r, Θ, Φ), where r is the radial distance from origin and Θ ranges from 0 to 2π and Φ ranges from 0 to π. The transformation relationship between the Cartesian and spherical coordinates of a point is defined as

$$x = r \sin \Phi \cos \Theta$$

$$y = r \sin \Phi \sin \Theta$$

$$z = r \cos \Theta.$$

Suppose A is a plane, R is a polygonal region in plane A, and P is a point not in plane A. The set of all segments that join P to a point of region R form a *pyramid* [9]. The region R is called the *base* of the pyramid and the point P is called the *top* of the pyramid. A pyramid is a *square pyramid* if the base is a square and the other faces than the base are congruent isosceles triangles. The square pyramid is defined by the length of the side of its base and by its height. A *diamond* is a polyhedron obtained by combining two equal square pyramids whose sides are equal to its height, as follows: other pyramid is turned upside down, and their bases are sticked together (see Figures 3.(a) and 4.(a)). The *base* of the diamond is the base of the pyramids and the *side* of the diamond is the side of the base of the pyramids. The *bottom* of the diamond is the top of the turned pyramid, and the *top* of the diamond is the top of the other pyramid. The *diameter* of the diamond is the height of the pyramid, i.e., the distance between the bottom and the top.

3 An algorithm for drawing series parallel-digraphs in three dimensions

In this section we introduce a new algorithm for drawing series parallel digraphs in three dimensional space. The main idea behind our drawing algorithm is to draw subdrawings inside three dimensional diamonds and then to combine these diamonds in such a way that the whole graph is drawn correctly. In the following, we assume that the decomposition (structure) tree is given as input.

Algorithm 1 recursively draws the given graph G inside a three dimensional diamond $\lozenge(G)$. A single edge is drawn inside a diamond having height 2. If G is a series composition of series-parallel digraphs G_1 and G_2, the drawings of G_1 and G_2 are placed one above the other inside a diamond having diameter $R_1 + R_2$ where $R_1 + R_2$ are the diameters of $\lozenge(G_1)$ and $\lozenge(G_2)$, respectively.

If G is a parallel composition of series-parallel digraphs G_1 and G_2, where G_2 is a Q-node, drawings of G_1 and G_2 are placed inside a diamond $\lozenge(G)$ having diameter $2R_1 + 2$, such that the bottom and the top of $\lozenge(G_2)$ are on the same line as the bottom and the top of $\lozenge(G)$, and the drawing of G_1 is placed inside $\lozenge(G)$ such that the projection of $\lozenge(G_1)$ on the xy-plane is in the down left corner of the projection of $\lozenge(G)$.

Notice that the binary decomposition tree assumed in the following algorithm allways exists as the result of the recognition algorithm introduced in [14].

Algorithm 1 $\Diamond - SP$

input: a series-parallel digraph G and its binary decomposition tree T
output: a three dimensional upward Fary grid drawing $\mathcal{D}$ of G

If G has only a single edge e
then e is drawn as a line of length 2 parallel to z-axis inside a diamond with diameter $R = 2$ (See Figures 3.(a) and 4.(a)).
If G is a series composition of G_1 and G_2
then draw recursively G_1 and G_2 to obtain drawings $\mathcal{D}_1$ and $\mathcal{D}_2$ with diameters R_1 and R_2. Identify the sink of G_1 with the source of G_2 and drawings $\mathcal{D}_1$ and $\mathcal{D}_2$ inside a diamond with diameter $R = R_1 + R_2$ such that the bottoms and tops of the $\mathcal{D}_1$ and $\mathcal{D}_2$ are placed on the same line (See Figures 3.(b) and 4.(b)).
If G is a parallel composition of G_1 and G_2
then draw recursively G_1 and G_2 to obtain drawings $\mathcal{D}_1$ and $\mathcal{D}_2$ with diameters R_1 and R_2.
If G_1 or G_2 consists of a single edge
then
Let $\mathcal{D}_2$ be the drawing of a single edge, and let $\mathcal{D}_1$ be the drawing of the other subgraph. Place $\mathcal{D}_1$ and $\mathcal{D}_2$ inside $\Diamond(G)$ with diameter $R = 2R_1 + 2$ such that the projection of $\mathcal{D}_1$ on the xy-plane is in the down left corner of the projection of $\Diamond(G)$ on the xy-plane and $\mathcal{D}_2$ is in the middle of $\Diamond(G)$ and that the bases of $\Diamond(G_1), \Diamond(G_2), \Diamond(G)$ lie on the same plane (See Figures 3.(c) and 4.(c)).
else
place $\mathcal{D}_1$ and $\mathcal{D}_2$ inside $\Diamond(G)$ with diameter $R_1 + R_2$ such that the projections on the xy-plane of $\Diamond(G_1)$ and $\Diamond(G_2)$ are in the down left corner and in the down-right corner of the projection of $\Diamond(G)$ on the xy-plane and that the bases of $\Diamond(G_1), \Diamond(G_2)$ and $\Diamond(G)$ lie on the same plane.
Identify the sources of G_1 and G_2 by moving them to the bottom of $\Diamond(G)$ and identify the sinks of G_1 and G_2 by moving them to the top of $\Diamond(G)$ (See Figures 3.(d) and 4.(d)).

If G_2 is not a Q-node, then the drawings of G_1 and G_2 are placed inside a diamond $\Diamond(G)$ having diameter $R_1 + R_2$ such that the projections on the xy-plane of $\Diamond(G_1)$ and $\Diamond(G_2)$ are in the down left corner and in the down right corner of the projection of $\Diamond(G)$ on the xy-plane and that the bases of $\Diamond(G_1), \Diamond(G_2)$ and $\Diamond(G)$ lie on the same plane. Figures 3 and 4 illustrate the series and parallel compositions.

Next we prove that Algorithm 1 works correctly and, after that, we investigate the volume of the obtained drawing.

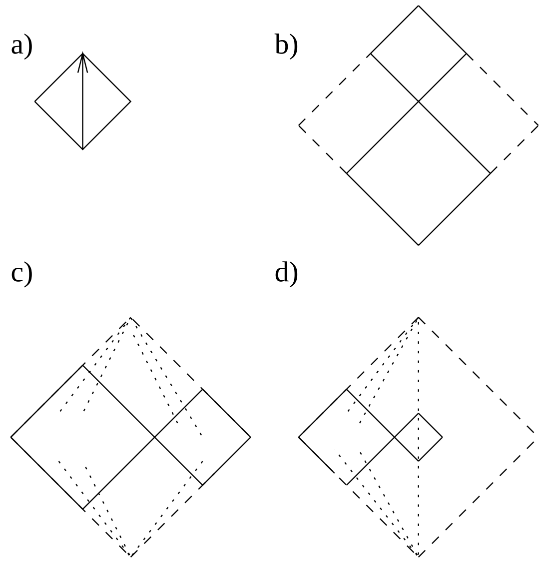

Figure 3: Illustration of $\Diamond$-SP Algorithm: (a) a digraph consisting of a single edge; (b) a series composition; (c) parallel composition; (d) a parallel composition with a transitive edge. Figures are projections on the yz-plane.

Theorem 3.1. *Let G be a series-parallel digraph with decomposition tree T. Algorithm $\Diamond$-SP produces a three dimensional strictly upward Fary grid drawing of G.*

Proof. The correctness of Algorithm 1 can be proved by showing that the following invariants hold after each parallel or series composition:

1. The drawing of G is contained inside a three dimensional diamond $\Diamond(G)$ having diameter R.
2. The source s is placed at the bottom of $\Diamond(G)$ and the sink t is placed at the top of $\Diamond(G)$.
3. For any vertex v adjacent to the source s of G, the spherical wedge $r < 0, 3\pi/4 \leq \Theta \leq 5\pi/4, 0 \leq \Phi \leq \pi/2$ contains only s.
4. For any vertex adjacent v to the sink t of G, the spherical wedge $r > 0, 3\pi/4 \leq \Theta \leq 5\pi/4, 0 \leq \Phi \leq \pi/2$ contains only t.
5. For the top of $\Diamond(G)$ the spherical wedge $-\pi/4 \leq \Theta \leq \pi/4$ contains all vertices and edges of G.

The proof is by induction on the vertices in the decomposition tree T. In each step we prove that each invariant holds after a series composition or after a parallel composition. Invariants 1 and 2 guarantee that the drawing is placed inside $\Diamond$, and the last three invariants guarantee that the vertices s and t can be moved to the top and to the bottom of the new drawing without creating crossings.

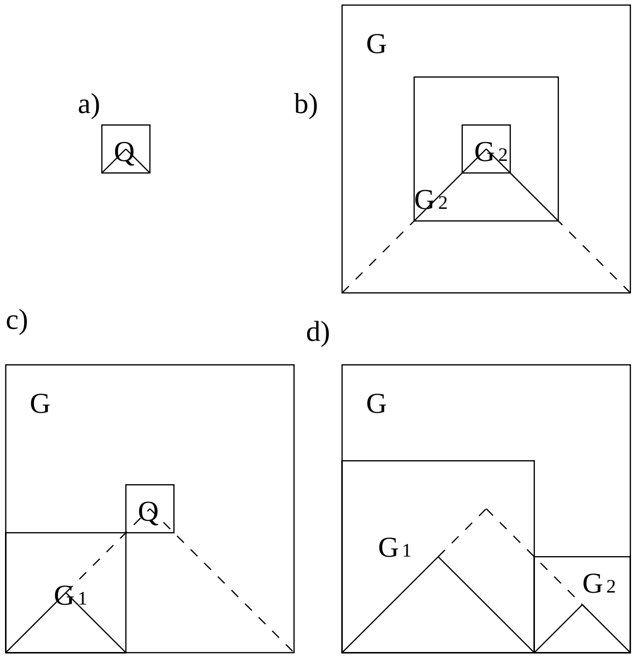

Figure 4: Illustration of $\Diamond$-SP Algorithm: (a) a digraph consisting of a single edge; (b) a series composition; (c) parallel composition; (d) a parallel composition with a transitive edge. Figures are projections on the xy-plane

If T has one node, then this node is Q which is associated to edge (v, u). Clearly all five invariants hold.

Suppose first that the node in the decomposition tree is of type S, i.e., G is the series composition of digraphs G_1 and G_2, with the drawings $\mathcal{D}_1$ and $\mathcal{D}_2$. Since series-parallel digraphs G_1 and G_2 are drawn correctly by the induction hypothesis, the invariants 1 and 2 are satisfied by the construction of algorithm. Also invariants $3-5$ are satisfied since the relative positions of the vertices of G_1 and G_2 are not changed.

Suppose next that the node in the decomposition tree is is of type P, i.e., G is the parallel composition of digraphs G_1 and G_2, with the drawings $\mathcal{D}_1$ and $\mathcal{D}_2$.

Now we have two different cases, depending on graph G_2 (If G_1 is a Q-node, we can swap G_1 and G_2 to have that G_2 is a Q-node).

Case 1: If G_2 is a transitive edge, then it is drawn in the middle of $\Diamond(G)$ having diameter $2R_1 + 2$, and G_1 is drawn such that the projection of $\Diamond(G_1)$ on the xy-plane is in the down left corner of the projection of $\Diamond(G)$. Also the bases of $\Diamond(G_1), \Diamond(G_2)$ and $\Diamond(G)$ lie on the same plane.

By the construction of algorithm, invariants 1 and 2 hold for the drawing of the graph G.

To see that invariant 5 holds, make projections of $\Diamond(G_1)$ and $\Diamond(G_2)$ and spherical wedge $-\pi/4 \leq \Theta \leq \pi/4$ from the top of $\Diamond(G)$ on the xy-plane. Now the projection of the wedge contains the projections of $\Diamond(G_1)$ and $\Diamond(G_2)$.

Invariants 3 and 5 for G_1 imply that the source s' of G_1 can be moved to the bottom of the $\Diamond(G)$ without creating crossings (see [1] for exact reasoning for the two dimensional case), and hence invariant 3 for G holds. Invariant 4 can be proved with a similar reasoning.

Case 2: If G_2 is not a transitive edge, then the drawings of G_1 and G_2 are placed inside a diamond $\Diamond(G)$ having diameter R_1+R_2 such that the projections of $\Diamond(G_1)$ and $\Diamond(G_2)$ on the xy-plane are in the down left corner and in the down right corner of the projection of $\Diamond(G)$ on the xy-plane and that the bases of $\Diamond(G_1), \Diamond(G_2)$ and $\Diamond(G)$ lie on the same plane.

By the construction of algorithm, invariants 1 and 2 hold for the drawing of the graph G.

To see that invariant 5 holds, make projections of the spherical wedge $-\pi/4 \leq \Theta \leq \pi/4$, $\Diamond(G_1)$ and $\Diamond(G_2)$ on the xy-plane. Now the projection of the wedge contains the projections of $\Diamond(G_1)$ and $\Diamond(G_2)$.

Invariants 1 and 5 for G_1 imply that the source s' of G_1 can be moved to the bottom of $\Diamond(G)$ without creating crossings (see [1] for exact reasoning for the two dimensional case), hence invariant 3 holds for G. Invariant 4 can be proved with a similar reasoning.

$\square$

Algorithm 2 describes a recursive method to calculate exact diameters for $\Diamond$ consisting the drawing of the graph to be drawn. If all diameters of nested diamonds are known, it is easy to compute the coordinates of vertices.

Algorithm 2 $\Diamond - SP - Label$

    input: the binary decomposition tree T of a series-parallel digraph G
    output: labeling of each subtree of T with diameter R

```
If the root of T is a Q-node
then
  R(T) = 2
else
  let T1 and T2 be the left and right subtrees of T, respectively
  for each i = 1,2 do
      ◊ − SP − Label(Ti)
  if the root of T is an S-node
  then
      R(T) = R(T1) + R(T2)
  else
      if G1 is a Q-node
      then
          swap G1 and G2 to have that G2 is a Q-node
      if G2 is a Q-node
      then
          R(T) = 2 * R(T1) + 2
      else
          R(T) = R(T1) + R(T2).
```

Next we prove an upper bound for the volume of the drawing produced by Algorithm 1.

Theorem 3.2. *Let G be a series-parallel digraph having m edges. Then Algorithm 1 produces a three dimensional strictly upward Fary grid drawing of G with volume $8m^3$.*

Proof. Let G be a series-parallel digraph consisting of series-parallel digraphs G_1 and G_2. By the construction of Algorithm 1, the height of the diamond $\Diamond(G)$ of G is at most $2(m_1 + m_2) = 2m$, where m_1 and m_2 are the diameters of $\Diamond(G_1)$ and $\Diamond(G_2)$, respectively.

Since the diameter of $\Diamond(G)$ is $2m$, the projection of $\Diamond(G)$ on the xy-plane is a square with sides of length $2m$ and parallel to x- and y-axis. Hence, the volume of the drawing is $2m \times 2m \times 2m = 8m^3$.

□

If the series-parallel digraph doesn't contain any transitive edges, we can prove a slightly better upper bound.

Theorem 3.3. *Let G be a series-parallel digraph having m edges such that there is no transitive edges. Then Algorithm 1 produces a three dimensional strictly upward Fary grid drawing of G with volume $m \times m \times m$.*

Proof. By the construction of Algorithm 1, the diameter of the diamond $\Diamond(G)$ is the sum of the diameters of the diamonds $\Diamond(G_1)$ and $\Diamond(G_1)$. Hence, the diameter of $\Diamond(G)$ is m and the volume of the drawing is $m \times m \times m$. □

4 Special cases

In this section we investigate improvements to the volume of drawing produced by Algorithm 1. We show that there are special cases when series-parallel digraphs can be drawn with lower volume than $O(n^3)$. We prove that if the series-parallel digraph G is regular, it can be drawn inside a box having volume $O(n^2)$. Moreover, if G is regular and the number of the children of a P-node in the decomposition tree T is always k^2, where $k \geq 2$, graph can be drawn inside a box having volume $O(n)$.

In the previous section we drawn components of the given series-parallel digraph inside a diamond. For regular series-parallel digraphs, we can draw components strictly inside a box which base is a square.

Next we give two simple lemmas, without proofs.

Lemma 4.1. *Let $S = \{p_1, \ldots, p_k\}$ be a set of points which are located on the plane L and let p be a point not in L. Then the set of lines $\{(p_1, p), \ldots, (p_k, p)\}$ do not intersect except at point p.*

Lemma 4.2. *[5, 10, 11] Let $S = \{s_1, \ldots, s_k\}$ be a set of equal squares, and let t be the side of a square. Then all squares from set S can be packed, with sides parallel to x- and y-axis, inside a square of side lt, where $(l-1)^2 < k \leq l^2$ and $l \in \mathbf{Z}_+$. The equality holds, if $\sqrt{k} = l$.*

If a series-parallel digraph is regular, it can be drawn such that all the adjacent vertices of the source (sink) lie on the same plane. This property yields that the source (sink) can be moved upward (downward) in any direction without creating crossings (Lemma 4.1).

We can now prove that regular series-parallel digraphs can be drawn with volume $O(n^2)$.

Algorithm 3 $\Box - regular - SP$

input: a regular series-parallel digraph G with structure tree T
output: a three dimensional strictly upward Fary grid drawing $\mathcal{D}$ of G

If G has only a single edge e
then e is drawn as a line of length 1 parallel to z-axis inside a box having dimension $1 \times 1 \times 1$.
If G is a series composition of digraphs $G_1, \ldots, G_k$
then draw recursively $G_1, \ldots, G_k$ to obtain drawings $\mathcal{D}_1, \ldots \mathcal{D}_k$ having dimensions $x \times x \times z$. Identify the sink of G_i with the source of G_{i+1}, where $1 \leq i \leq k-1$ and place drawings inside a box having dimension $x \times x \times zk$.
If G is a parallel composition of $G_1, \ldots, G_k$
then draw recursively $G_1, \ldots, G_k$ to obtain drawings $\mathcal{D}_1, \ldots, \mathcal{D}_k$ having dimensions $x \times x \times z$. Place the drawings inside a box having dimensions $lx \times lx \times z$, where $(l-1)^2 < k \leq l^2$. Identify the sources of $G_1, \ldots, G_k$ by moving them to the any point of the bottom-side of the box and identify the sinks of $G_1, \ldots, G_k$ by moving them to the any point of the top-side of the box.

Theorem 4.1. *Let G be a regular series-parallel digraph having n nodes and let T be the structure tree for G. Then Algorithm 3 produces a three dimensional strictly upward Fary grid drawing of graph G with volume $O(n^2)$.*

Proof. The proof is by induction on vertices in the structure tree T. We prove that the following invariants are satisfied after each series or a parallel composition:

1. The drawing of G is contained inside a three dimensional box $\Box(G)$ whose bottom is a square.
2. The source s is placed at the bottom of $\Box(G)$ and the sink t is placed at the top of $\Box(G)$.
3. For all vertices v adjacent to the source s of G, vertices lie on the same plane parallel to the xy-plane.
4. For all vertices v adjacent to the sink t of G, vertices lie on the same plane parallel to the xy-plane.
5. The volume of $\Box(G)$ is $O(m^2)$.

If T has one node, then this node is of type Q. The corresponding single edge can be drawn inside a box having dimensions $1 \times 1 \times 1 = O(m^2)$.

Case 1: Suppose that the node in the structure tree is of type S having k children. Note that all the children of S-node are P-nodes or Q-nodes since G is regular.

By the induction hypothesis, the common volume of the identical subdrawings $\mathcal{D}_1, \ldots, \mathcal{D}_k$ associated to the children of the S-node is $O(m^2)$, where m is the common number of edges in the corresponding subgraphs. The number of edges in G is km. Let $x \times x \times z$ be the dimensions of subdrawing $\mathcal{D}_1$. Since the series-parallel digraph is regular, all subdrawings can be drawn as a copy of the first subdrawing, so we have the same dimensions for all subdrawings.

Invariants 3 and 4 are satisfied since the relative positions of the vertices of G_1 and G_k are not changed.

To prove that invariant 5 is satisfied, place boxes containing these drawings one on the other. The drawing so achieved has volume $x \times x \times zk = k(x \times x \times z) = kO(m^2) = O(m^2)$.

Case 2: Suppose that the node in the structure tree is of type P with k children. Note that all the children of a P-node are S-nodes, since parallel edges are not allowed and a single Q-node contradicts regularity.

By the induction hypothesis, the common volume of the identical subdrawings $\mathcal{D}_1, \ldots, \mathcal{D}_k$ associated to the children of the P-node is $O(m^2)$, where m is the common number of edges in the corresponding subgraphs. The number of edges in G is km. Let $x \times x \times z$ be the dimensions of $\mathcal{D}_1$. Since G is regular, all subdrawings can be drawn as a copy of the first subdrawing.

By Lemma 4.2, these squares can be packed inside a box having footprint l^2, where $(l-1)^2 < k \leq l^2$. Let A' be total area of $\mathcal{D}_1, \ldots, \mathcal{D}_k$. For the footprint A of the box containing all these subdrawing, we have $A \leq 2A'$.

By Lemma 4.1, we can find points s and t from the bottom and from the top of the box such that there doesn't appear any crossings if nodes $s_1, \ldots, s_k$ are associated to s and $t_1, \ldots, t_k$ are associated to t. After this the drawing has still the property that all adjacent vertices to s and t are located on the same plane.

The footprint of the box containing all subdrawings is less than twice the total area of all subdrawings and the height of the drawing is the height of the subdrawings. Thus, the volume of the box is $O(m^2) = O(n^2)$. The theorem follows.

$\square$

If the parallel composition can be implemented without wasting any space when boxes are placed to the plane, it is possible to improve the result for the regular series-parallel digraphs even further. If P-node has always a suitable number of children, their drawings can be combined more efficiently. Next we show that if series-parallel digraph G is regular and if each P-node of its structure tree T has k^2 children where $k \geq 2$, then G can be drawn with volume $O(n)$. Since this is a special case of the Theorem 4.1, we only only sketch the proof.

Theorem 4.2. *Let $G = (V, E)$ be a regular series-parallel digraph having n nodes. Further, let T be the structure tree for G with the property that each P-node has k^2 children, where $k \geq 2$. Then G allow a three dimensional strictly upward Fary grid drawing with volume $O(n)$.*

Proof. The increase of the volume for regular series-parallel digraphs is linear in series-composition. Also the increase of the volume in parallel composition is linear if each P-node has k^2 children, since the subdrawings of P-node can be drawn inside a square without wasting any space (by Lemma 4.2). Thus the volume of the drawing is $O(n)$. $\square$

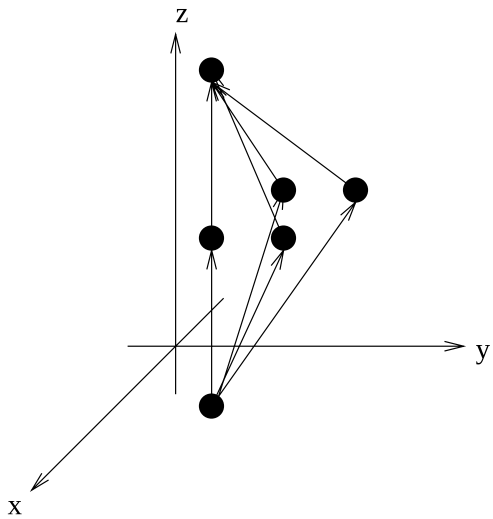

Figure 5: A three dimensional drawing obtained from the series-parallel digraph in Figure 1 using Algorithm 2 modified for the special case shown in Theorem 4.2.

5 Conclusions

In this paper we have proved that series-parallel digraphs can be drawn to three dimensional space inside a box having volume $O(n^3)$. We have also shown that if a series-parallel digraph is regular, then it can be drawn with volume $O(n^2)$. Also this result can be improved, if the series-parallel digraph is regular and each P-node in the corresponding structure tree has $k^2, k \geq 2$ children, when it is possible to draw inside a box having volume $O(n)$.

It is still an open question, if an arbitrary series-parallel digraph can be drawn with volume less that $O(n^3)$.

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